ASSOUAD NAGATA DIMENSION ON THE INTEGERS

1. Basic Facts and Definitions

Definition 1.1. A metric space X is said to be of *Assouad-Nagata dimension* $\leq n$, i.e. $dim_{AN} X \leq n$, iff there exists a constant c so that $\forall\ r > 0$ we can construct a cover $\mathfrak{U} = \{U_0, U_1, \dots, U_n\}$ of X such that each U_i is r-disjoint and uniformly bounded by cr.

Definition 1.2. A metric space X has *Assouad-Nagata dimension* $= n$ if $dim_{AN} X \leq n$ is true and $dim_{AN} X \leq n-1$ is false.

Remark 1.3. *Three points:*

(i) *Each U_i is a family of sets.*
(ii) *r-disjoint means that any two disjoint sets in U_i have distance at least r.*
(iii) *A family of sets is called uniformly bounded by D if every set of the family has diameter less than D*

Definition 1.4. For different values of r the map $D_x^n : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $D_x^n(r) = cr$ which gives the bound of the diameters is formally called an n-dimensional control function. In Assouad Nagata dimension the n-dimensional control function is a dilation.

Definition 1.5. A *bi-Lipschitz* function $f : (X, d_X) \to (Y, d_Y)$ satisfies the property

$$\mu d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y)$$

for some $\mu, \lambda > 0$.

Proposition 1.6. *Suppose that X and Y are metric spaces and that $dim_{AN} X \leq n$. If $f : X \to Y$ is bi-Lipschitz and onto then $dim_{AN} Y \leq n$.*

Let U a covering of X. Use $f(U)$ to create a covering of Y.

Remark 1.7. *dim_{AN} is bi-Lipschitz invariant.*

Proposition 1.8. *For a metric space (X, d), the following are equivalent:*

(i) *$dim_{AN} X \leq n$*
(ii) *$\exists\ c_1$ such that $\forall\ r > 0$, $\exists$ a cover U_r of X with r-multiplicity at most $n+1$ and mesh at most $c_1 r$.*
(iii) *$\exists\ c_2 > 0$ such that $\forall r > 0$, $\exists$ a cover V_r of X with multiplicity at most $n+1$, mesh at most $c_2 r$ and Lebesque number at least r.*

For the proof of the above see [7]

Definition 1.9.

- *$mesh(U_r) = sup\{diam(A) | A \in U_R\}$*
- *$multiplicity(U_r) = max\{\#Ai \in U_r,$ such that $\cap_i A_i \neq \emptyset\}$*

Date: April 1, 2008.
Key words and phrases. Assouad-Nagata dimension.

- $r-multiplicity(U_r) = max\{\#A_i$ intersecting a ball, $B_r(x)\}$
- $Lebesque\ number(U_r) = inf\{sup\{dist(x, X-U) : U \in U_r\} : x \in X\}$

Proposition 1.10. *If $X = A \cup B$, then $dim_{AN}X \leq max\{dim_{AN}A, dim_{AN}B\}$.*

The proof is really straight forward.

Definition 1.11. A metric space X is said to be of *asymptotic Assouad-Nagata dimension* n, i.e. $asdim_{AN}X = n$, if n is the smallest natural number for which $\exists\ c, d$ constants such that $\forall r > 0$ there exists a cover $U = \{A_0, A_1, \ldots, A_n\}$ of X such that each A_i is a family of r-disjoint sets and $\text{diam}(A_{i_j}) \leq cr + d$,

Remark 1.12. *In asymptotic Assouad-Nagata dimension the dimension control function is linear and not just a dilation.*

Proposition 1.13. *For a discrete metric space X, $asdim_{AN}X = dim_{AN}X$. In particular, if X is a finitely generated group with the word metric, then $asdim_{AN}X = dim_{AN}X$.*

For a detailed proof of the above see [5]

2. Finitely Generated Groups as Metric Spaces

Definition 2.1. Let $G =< S|R >$, $|S| < \infty$. Define the following word norm on G:

$$||x|| = min\{l(w) \quad x = w = s_{i_1}s_{i_2}\ldots s_{i_k} \quad s_{i_j} \in S\}$$

where $k = l(w)$ is the length of the word w.

Definition 2.2. Define the *word metric* on G by $d_w(x, y) = ||x^{-1}y||$.

Proposition 2.3. *d_w is left G-invariant, $d_w(x, y) = d_w(gx, gy)$.*

Proof.

$$d_w(gx, gy) = ||(gx)^{-1}gy|| = ||x^{-1}g^{-1}gy|| = ||x^{-1}y|| = d_w(x, y)$$

□

Proposition 2.4. *(G, d_w) is a discrete metric space.*

Proof. $d_w(x, y) \in \mathbb{Z}$. □

Corollary 2.5. *With d_w, $asdim_{AN}G = dim_{AN}G$.*

Proposition 2.6. *If G is finitely generated and $diam(G) < M$ then G is finite.*

Proof. For any $g \in G$, $||g|| \leq M$. If $|S| = k$ then $|G| \leq (2k)^M$. □

Proposition 2.7. *If G is finitely generated and $diam(G) < M$ then $asdim_{AN}G = 0$.*

Proof. Let $U_0 = G$. Consider $c = 1$, $d = M$. Then for all $r > 0$, U_0 is a cover of G, U_0 is trivially r-disjoint, and $diam(U_0) \leq D^n_x(r) = 1r + M$. □

Corollary 2.8. *If G is finitely generated and $diam(G) < M$ then $dim_{AN}G = 0$. See 1.13.*

Proposition 2.9. *If G is finite then $asdim_{AN}G = dim_{AN}G = 0$.*

Proof. Let $A_0 = G$. □

Proposition 2.10. *If $asdim_{AN} G = 0$, and G finitely generated, then G is finite.*

Proof. Let $r \geq 1$ and consider the appropriate covering. Let $G \subset U_0 = \{A_0, A_1, \ldots\}$. $|A_0| = k < \infty$ because, $diam(A_0) \leq c1 + d = c + d$. As a result, if $A_0 \supset G$, then G is finite, as desired. Assuming that this is not the case, there exist $x \in A_0$ and $s \in S \cup S^{-1}$ such that $xs \notin A_0$. If $xs \in A_n$, then we have that A_0 and A_n are at most 1-disjoint, contradiction. Thus, $U_0 = \{A_0\}$ and G is finite. □

Definition 2.11. A map $\| \cdot \|_G : G \to \mathbb{R}$ is said to be a *proper norm* if it satisfies the following conditions:

i) $\|g\|_G = 0$ if and only if g is the neutral element in G.
ii) $\|g\|_G = \|g^{-1}\|_G$ for every $g \in G$.
iii) $\|g \cdot h\|_G \leq \|g\|_G + \|h\|_G$ for all $g, h \in G$
iv) For every $k > 0$ the number of elements in G such that $\|g\|_G \leq k$ is finite.

Proposition 2.12. *Suppose G is an infinite, finitely generated group with a proper norm. Consider $d : G \times G \to \mathbb{R}^+$ the induced metric. Then $asdim_{AN}(G, d) > 0$.*

Proof. Assume not. Let

$$k = min\{||s_i||; s_i \in S \cup S^{-1}\}$$

Let $r = k + 1$. Then there exists a cover of G containing only one family of sets, $\mathfrak{U} = \{A\}$, with $A = \{A_1, A_2, \ldots\}$. Suppose $A_i \supset G$ for some i. Then $diam(G) \leq diam(A_i) \leq D$. Since G is discrete G has to be finite. Contradiction. Thus G is not fully contained in any of the A_i's. In particular G is not contained in A_1. So there exists some $x \in A_1$ with $xs \notin A_1$ where $s \in (S \cup S^{-1})$. Then, if $xs \in A_j$, A_1 and A_j are at most k-disjoint, and so they are not r-disjoint. Contradiction. □

Remark 2.13. *Since $\mathbb{Z} =< 1 >$ is an infinite, free group, with one generator, from the proposition above we have that $asdim_{AN}(\mathbb{Z}, d_w) \neq 0$. Further, we would like to know precisely the $asdim_{AN}\mathbb{Z}$. That result is an immediate corollary of the following theorem.*

Theorem 2.14. *If T is a tree then $asdim_{AN} T \leq 1$.*

Proof. Fix e to be the origin of the tree. Let $d > 0$. Consider $A_k = \{x \in T : kd \leq d(x, e) < (k + 1)d\}$ the "Annulus" of radius $(kd, (k + 1)d)$ for all $k \in \mathbb{N}$ Consider the Gromov product on the tree with e as base point:

$$(x|y) = \frac{1}{2}(d(x, e) + d(y, e) - d(x, y))$$

Define the relation:

$$x \sim y \Longleftrightarrow (x|y) \geq (k - \frac{1}{2})d$$

Recall that the Gromov product on the tree satisfies the relation:

$$(x|y) \geq min\{(x|z), (y|z)\}$$

since the tree is 0-hyperbolic. This proves that the relation is transitive. The other two properties are trivial thus we have an equivalence relation. Let $U_{i,k}$ the equivalence classes in A_k. Name:

$$\begin{aligned} \mathfrak{U}_0 &= \{U_{i,k}, i \in I_k, k \in \mathbb{N} \text{ even}\} \\ \mathfrak{U}_1 &= \{U_{j,k}, j \in J_k, k \in \mathbb{N} \text{ odd}\} \end{aligned}$$

□

These families are d-disjoined. We will prove it for $\mathfrak{U}_0$ and the proof is similar for $\mathfrak{U}_1$. Let $U_{1,k}$, $U_{2,l}$ be two sets of $\mathfrak{U}_0$. If $k \neq l$ then clearly

$$d(U_{1,k}, U_{2,l}) \geq d(A_k, A_l) = |k-l|d$$

and since $k \neq l$ we have $|k-l| \geq 1$ since both are integers. Thus $d(U_{1,k}, U_{2,l}) \geq d$. If $k = l$ then since $U_{1,k}$ and $U_{2,l}$ are different components of the relation we have the following: For every $x \in U_{1,k}$ and $y \in U_{2,l}$ $(x|y) < (k - \frac{1}{2})d)$ thus

$$-2(x|y) > -2kd + d$$

so:

$$\begin{aligned} d(x,y) &= d(x,e) + d(y,e) - 2(x|y) \\ &> kd + kd - 2kd + d \\ &= d \end{aligned}$$

This proves that $U_{1,k}$ and $U_{2,l}$ are d-disjoined. They are also $3d$-bounded. Let $x, y \in U_{1,k}$. Then $(x|y) \geq (k - \frac{1}{2})d \Rightarrow -2(x|y) \leq -2kd + d$ but then $d(x,y) = d(x,e) + d(e,y) - 2(x|y) \leq (k+1)d + (k+1)d - 2kd + d = 3d$. Clearly the two families cover the tree so from the definition of the Assouad-Nagata dimension we have that $asdim_{AN} \leq 1$.

Corollary 2.15. *With d_w, $asdim_{AN}\mathbb{Z} = 1$*

Proof. By the previous proposition, $asdim_{AN}(\mathbb{Z}, d_w) \geq 1$ and by our theorem, $asdim_{AN}(\mathbb{Z}, d_w) \leq 1$ so $asdim_{AN}(\mathbb{Z}, d_w) = 1$. ☐

3. Monotone norms on $\mathbb{Z}$

Remark 3.1. *Thus far, we have restricted our investigation to the norm that is most natural to apply to finitely generated groups. However, this restriction has been self-imposed and there is no technical reason to avoid other types of norms. We now generalize our results for proper, monotone norms.*

Lemma 3.2. *If $p : \mathbb{R} \to \mathbb{R}^+$ is an increasing function then $B_1(x, p(t)) = B_2(x, t)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ where $B_1(x, p(t)) = \{y \in \mathbb{Z} : p(x-y) \leq p(r)\}$ and $B_2(x,t) = \{y \in \mathbb{Z} : |x-y| \leq r\}$.*

Proof. Notice that $y \in B_1(x, p(r)) \Leftrightarrow p(x-y) \leq p(r)$. Since p is increasing we get $p(x-y) \leq p(r) \Leftrightarrow |x-y| \leq r$. Clearly $|x-y| \leq r \Leftrightarrow y \in B_2(x,r)$. Combining the above we have the wanted inclusions.

☐

Lemma 3.3. *If $p : \mathbb{R} \to \mathbb{R}^+$ is increasing then the families:*

$$\begin{aligned} \mathfrak{U}_0 &= \{B_1(4kt, p(t)) : k \in \mathbb{Z}\} \\ \mathfrak{U}_1 &= \{B_1((4k+2)t, p(t)) : k \in \mathbb{Z}\} \end{aligned}$$

are $p(t)$-disjoined and $2p(t)$-bounded, where $B_1(x,s)$ as the previous lemma. Finally, they form a cover of $\mathbb{Z}$.

Proof. From the previous lemma we have that:

$$\begin{aligned} \mathfrak{U}_0 &= \{B_2(4kt, t) : k \in \mathbb{Z}\} \\ \mathfrak{U}_1 &= \{B_2((4k+2)t, t) : k \in \mathbb{Z}\} \end{aligned}$$

Clearly $\mathfrak{U}_0$ and $\mathfrak{U}_1$ cover $\mathbb{Z}$. Given $a \in \mathbb{Z}$, there exists an integer k such that $4kt \leq a < 4(k+1)t$. Then, either $4kt \leq a < 4kt+t$ and $a \in B_2(4kt, t)$, $(4k+2)t-t \leq a < (4k+2)t+t$ and $a \in B_2((4k+2)t, t)$ or $4(k+1)t - t \leq a < 4(k+1)t$ so $a \in B_2(4(k+1)t, t)$.

Also they are $2p(t)$-bounded since:

$$diam(B_1(4kt, p(t))) \leq 2p(t)$$

Finally it is easy to show that $\mathfrak{U}_0$ and $\mathfrak{U}_1$ are $p(t)$-disjoined. We will prove it for $\mathfrak{U}_0$ and the proof is the same for $\mathfrak{U}_1$. Let $B_1(4k_1t, p(t))$ and $B_1(4k_2t, p(t))$ with $k_1 \neq k_2$ such that they are not $p(t)$-disjoined. Then, since both of them are compact, there exist $x \in B_1(4k_1t, p(t))$ and $y \in B_1(4k_2t, p(t))$ such that $p(|x-y|) < p(t)$. Since p is increasing we get $|x - y| < t$. But then

$$|4k_1t - 4k_2t| \leq |4k_1t - x| + |x - y| + |y - 4k_2t| \qquad \text{(A)}$$

Since $x \in B_1(4k_1t, p(t))$ we get $x \in B_2(4k_1t, t)$ and thus $|x - 4k_1t| \leq t$. Similarly $|y - 4k_2t| \leq t$. Thus (A) becomes:

$$|4k_1t - 4k_2t| \leq 3t$$

which leads to $|k_1 - k_2| \leq \frac{3}{4}$. Since $k_1, k_2 \in \mathbb{Z}$ we have $k_1 - k_2 = 0$ and thus $k_1 = k_2$ which is a contradiction. Thus $\mathfrak{U}_0$ is $p(t)$-disjoined.

□

Proposition 3.4. *Suppose $p : \mathbb{R} \to \mathbb{R}^+$ is an increasing, proper, non-bounded norm. Consider the induced metric $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$:*

$$d(x, y) = p(|x - y|)$$

Then $asdim_{AN}(\mathbb{Z}, d) = 1$

Proof. Let $r > 0$. Since p is not bounded there exists an $a \in \mathbb{Z}$ such that $p(a-1) \leq r$ and $p(a) \geq r$. Consider the two families

$$\begin{aligned} \mathfrak{U}_0 &= \{B_1(4ka, p(a)) : k \in \mathbb{Z}\} \\ \mathfrak{U}_1 &= \{B_1((4k+2)a, p(a)) : k \in \mathbb{Z}\} \end{aligned}$$

From the previous lemma we have that the $\mathfrak{U}_0, \mathfrak{U}_1$ are a covering of $\mathbb{Z}$. Also they are $p(a)$-disjoined. Since $p(a) \geq r$ they are also r-disjoined. Finally they are $2p(a)$-bounded. Notice that

$$2p(a) \leq 2(p(a-1) + p(1)) \leq 2p(a-1) + 2p(1) \leq 2r + 2p(1)$$

name $c = 2$, $d = 2p(1)$. Then the families are $cr + d$ disjoined. Since r was chosen arbitrarily and c, d are constants from the definition of asymptotic Assouad - Nagata dimension we have $asdim_{AN}(\mathbb{Z}, d) \leq 1$.

From a previous proposition([2.12]) $asdim_{AN}(\mathbb{Z}, d) \geq 1$ which concludes the proof.

□

Proposition 3.5. *Suppose that $p : \mathbb{R} \to \mathbb{R}^+$ is a decreasing norm. Then the induced metric on $\mathbb{Z}$ is not proper.*

Proof. Suppose that the induced metric was proper. Consider the ball $B(x, p(r)) = \{y \in \mathbb{Z} : p(x - y) < p(r)\}$. Since p is decreasing we have that $B(x, p(r)) \supseteq A = \{y \in \mathbb{Z} : |x - y| > r\}$. But $B_1(x, p(r))$ is compact. Since it is a metric space

$B_1(x, p(r))$ is bounded and thus finite since the metric is proper and $\mathbb{Z}$ is discrete. On the other hand A is clearly infinite. Contradiction. □

Remark 3.6. *Since we only deal with proper norms we can now say that our norm is monotone and mean that it is increasing.*

Corollary 3.7. *Suppose that $p : \mathbb{R} \to R^+$ is a norm which is constant in $\mathbb{Z}$ (and 0 at 0). Then the induced metric is not proper.*

Proof. Let $p(x) = d$ for all $x \neq 0$ in $\mathbb{Z}$. Then the ball $B(0, 2d) = \mathbb{Z}$, is infinite and thus the norm cannot be proper. □

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